8.2 – Compositions and Inverse Transformations

Definition: If $T : V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be **one-to-one** if T maps distinct vectors in V into distinct vectors in W.
OR $T : V \to W$ is one-to-one if and only if for each vector $\mathbf{w}$ in the range of T, there is exactly one vector $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$.
OR $T : V \to W$ is one-to-one if and only if $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$.

Definition: If $T : V \to W$ is a linear transformation from a vector space V to a vector space W, then T is said to be **onto** (or **onto W**) if every vector in W is the image of at least one vector in V.

Theorem 8.2.1 If $T : V \to W$ is a linear transformation, then T is one-to-one if and only if $\ker(T) = \{\mathbf{0}\}$.

#4 Determine whether the linear transformation is one-to-one by finding its kernel and then applying Theorem 8.2.1.

a. $T : R^3 \to R^3$, where $T(x,\ y) = (x - y,\ y - x,\ 2x - 2y)$

b. $T : R^2 \to R^2$, where $T(x,\ y) = (0,\ 2x + 3y)$

c. $T : R^2 \to R^2$, where $T(x,\ y) = (x + y,\ x - y)$

Theorem 8.2.X If $T : V \rightarrow W$ is a one-to-one linear transformation and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a linearly independent subset of V, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ is linearly independent.

#5 Determine whether multiplication by A is one-to-one by computing the nullity of A and then applying Theorem 8.2.1

a. $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -3 & 6 \end{bmatrix}$ b. $A = \begin{bmatrix} 1 & 3 & 1 & 7 \\ 2 & 7 & 2 & 4 \\ -1 & -3 & 0 & 0 \end{bmatrix}$

Theorem 8.2.2 If V and W are finite-dimensional vector spaces with the same dimension, and if $T : V \to W$ is a linear transformation, then the following statements are equivalent.
a) T is one-to-one.
b) $\ker(T) = \{\mathbf{0}\}$.
c) T is onto [i.e., $R(T) = W$].

Theorem 8.2.3 If $T_A : R^n \to R^m$ is a matrix transformation, then
a) T_A is one-to-one if and only if the columns of A are linearly independent.
b) T_A is onto if and only if the columns of A span R^m.

#14 Use Theorem 8.2.3 to determine whether multiplication by A is one-to-one, onto, both, or neither. Justify your answer.

a. $A = \begin{bmatrix} 9 & -3 \\ -4 & 2 \\ 1 & 1 \end{bmatrix}$ b. $A = \begin{bmatrix} 3 & -3 & 1 & 1 \\ 6 & -6 & 0 & 2 \\ 9 & -9 & 1 & 3 \end{bmatrix}$

c. $A = \begin{bmatrix} 3 & -9 \\ -1 & 3 \end{bmatrix}$

d. $A = \begin{bmatrix} 2 & 3 & 8 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

Theorem 8.2.4 Equivalent Statements (extends Theorems 4.9.8, 5.1.5 and 6.4.5, the last two of which we have not seen)

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible.

b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n.

d) A is expressible as a product of elementary matrices.

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

g) $\det(A) \neq 0$.

h) The column vectors of A are distinct and linearly independent.

i) The row vectors of A are distinct and linearly independent.

j) The column vectors of A span R^n.
k) The row vectors of A span R^n.
l) The column vectors of A form a basis for R^n.
m) The row vectors of A form a basis for R^n.
n) A has rank n.
o) A has nullity 0.
p) The orthogonal complement of the null space of A is R^n.
q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.
r) $\lambda = 0$ is not an eigenvalue of A. [This is from Theorem 5.1.5.]
s) $A^T A$ is invertible. [This is from Theorem 6.4.5.]
t) The kernel of T_A is $\{\mathbf{0}\}$.
u) The range of T_A is R^n.
v) T_A is one-to-one.

Definition: If $T : V \to W$ is a one-to-one linear transformation with range $R(T)$, and if $\mathbf{w}$ is any vector in $R(T)$, then there is exactly one vector $\mathbf{v}$ in V for which $T(\mathbf{v}) = \mathbf{w}$. The **inverse of T** maps $\mathbf{w}$ back into $\mathbf{v}$ and is denoted by T^{-1}. (This is analogous to **inverse** in Section 1.9.)

Definition: (analogous to **composition** in Section 1.9)
If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then the **composition** of T_2 with T_1, denoted by $T_2 \circ T_1$ (which is read "T_2 circle T_1") is the function defined by the formula $(T_2 \circ T_1)(\mathbf{u}) = T_2(T_1(\mathbf{u}))$ where $\mathbf{u}$ is a vector in U.

Theorem 8.2.5 If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, then $(T_2 \circ T_1) : U \to W$ is also a linear transformation.

Theorem 8.2.6 If $T_1 : U \to V$ and $T_2 : V \to W$ are one-to-one linear transformations, then:
a) $T_2 \circ T_1$ is one-to-one.
b) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$.

#38 Let $D(\mathbf{f}) = f'(x)$ and $J(\mathbf{f}) = \int_0^x f(t)dt$ be the derivative and integral linear transformations. Find $(J \circ D)(\mathbf{f})$ for

a. $\mathbf{f}(x) = x^2 + 3x + 2$

b. $\mathbf{f}(x) = \sin x$

Key concepts from 8.3:

Definition 1: A linear transformation $T : V \to W$ that is both one-to-one and onto is said to be an **isomorphism**, and W is said to be **isomorphic** to V.

Theorem 8.3.1 Every real n-dimensional vector space is isomorphic to R^n. [In this sense, every n-dimensional vector space is algebraically equivalent to R^n.]

Theorem 8.3.2 If S is an ordered basis for a vector space V, then the coordinate map $\mathbf{u} \xrightarrow{T} (\mathbf{u})_S$ is an isomorphism between V and R^n.

Theorem 8.3.X A linear transformation $T : V \to W$ is an isomorphism if and only if whenever $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a basis for V, $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \dots, T(\mathbf{v}_k)\}$ is a basis for W.

Examples of Natural isomorphisms

Between P_{n-1} and R^n: $a_0 + a_1x + \dots + a_{n-1}x^{n-1} \xrightarrow{T} (a_0, a_1, \dots, a_{n-1})$

Between M_{22} and R^4: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{T} (a,\ b,\ c,\ d)$